UNIQUENESS IN CERTAIN INVERSE PROBLEMS OF THE THEORY OF HEAT CONDUCTION

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Uniqueness theorems are proved for inverse two-dimensional problems of the theory of heat conduction in two different formulations.

1. The problem of restoring the initial temperature field distribution over the section of a specimen by measurements performed on its surface in recent time is considered. Such a problem occurs particularly in studying the control problems of certain technological processes associated with the heat treatment of metallic articles, and corresponds to the possibilities of measurement techniques. The problem is among the class of inverse problems, for an appropriate formulation assuring uniqueness of the solution; this latter can be found by using the method of regularization [1], which is effective in a broad circle of heat-conduction problems [2, 3].

The present paper is devoted to clarifying sufficient conditions for the uniqueness of the solution of the problem formulated. The question of the restoration of the initial temperature distribution was examined in addition to others in [4] as well as in [5]. However, a given temperature distribution everywhere in a certain recent time corresponds to the problem formulations. Questions of uniqueness are not touched upon in [4, 5]. Questions of uniqueness were considered in addition to the problem of stability of the solution in [6, 7], where the initial temperature distribution was desired in [6].

Similarly to [6, 7], we use the minimal experimentally accessible initial information and, in contrast to [6], for the solution of the two-dimensional problem. Its general mathematical formulation can be given as follows: find the solution of the operator equation $A\varphi = U$, where $\varphi(M)$, $M \in \Omega$ is the initial temperature field in the domain Ω; $U = U(P, t)$, $P \in \partial\Omega$, $0 < t_1 \leqslant t \leqslant t_2$ is the measured temperature on a certain part of the specimen surface in the time interval $[t_1, t_2]$, and A is a linear integral operator. The uniqueness is analyzed below for two specific formulations of the problems.

2. Let us consider the heat-conduction equation in the two-dimensional case in a Cartesian coordinate system:

$$U_t = a^2 (U_{xx} + U_{yy}),\ x,\ y \in \Omega,\ \Omega\colon\ 0 < x < l_1,\ 0 < y < l_2; \qquad U(x,\ y,\ 0) = U_0(x,\ y) \tag{1}$$

for convective heat transfer from a medium of zero temperature according to Newton's law:

$$(U_x + hU)|_{x=l_1} = 0;\ (U_x - hU)|_{x=0} = 0;\ (U_y + hU)|_{y=l_2} = 0; \qquad (U_y - hU)|_{y=0} = 0. \tag{2}$$

The problem formulated corresponds to cooling of a rod of rectangular section, of infinite length in the 0z axis direction. We assume that measurement of the temperature of part of the specimen surface is possible during a certain time interval, i.e., the following function is known

$$U(x,\ 0,\ t) = \varphi(x,\ t),\ 0 < t_1 \leqslant t \leqslant t_2,\ 0 \leqslant x \leqslant l_1. \tag{3}$$

Let us prove that a unique determination of the initial distribution $U_0(x, y)$ is possible according to this function $\varphi(x, t)$. The solution of problem (1)-(2) is represented in the form [8]

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$$U(x, y, t) = \sum_{m,n=1}^{\infty} A_{mn} \exp(-a^2(\lambda_m^2 + \mu_n^2) t) X_m(x) Y_n(y),$$

$$X_m(x) = \sin(\lambda_m x + \omega_m), \; Y_n(y) = \sin(\mu_n y + \omega_n), \tag{4}$$

$$\mathrm{tg}\, \omega_m = \lambda_m/h, \; 0 < \omega_m < \pi/2, \; \mathrm{tg}\, \omega_n = \mu_n/h, \; 0 < \omega_n < \pi/2.$$

The values of λ_m and μ_n are determined as follows

$$\mathrm{ctg}(\lambda_m l_1) = 0.5(\lambda_m/h - h/\lambda_m), \; \mathrm{ctg}(\mu_n l_2) = 0.5(\mu_n/h - h/\mu_n).$$

Therefore, the problem reduces to proving the unique determination of the coefficients A_{mn} according to the known function $\varphi(x, t)$:

$$\varphi(x, t) = \sum_{m,n=1}^{\infty} A_{mn} \exp(-a^2(\lambda_m^2 + \mu_n^2) t) \sin(\lambda_m x + \omega_m) \sin \omega_n, \tag{5}$$

$$0 \leqslant x \leqslant l_1, \; t_1 \leqslant t \leqslant t_2.$$

Let us convolute the right and left sides of (5) with $X_m(x)$

$$\int_0^{l_1} \varphi(x, t) \sin(\lambda_m x + \omega_m)\, dx = \sum_{n=1}^{\infty} A_{mn} \exp(-a^2(\lambda_m^2 + \mu_n^2) t) \sin \omega_n \, \|X_m\|_{L_2}^2, \tag{6}$$

where m is a fixed number. Or in another form

$$\psi_m(t) = \sum_{n=1}^{\infty} A_{mn} \exp(-a^2(\lambda_m^2 + \mu_n^2) t) \sin \omega_n,$$

$$\psi_m(t) = \frac{\int_0^{l_1} \varphi(x, t) \sin(\lambda_m x + \omega_m)\, dx}{\|X_m(x)\|_{L_2}^2}.$$

Let us examine the series obtained for fixed m, we prove its uniform convergence on the half-axis $[0, \infty]$. To do this, we estimate A_{mn}. We execute the estimates in the class C_4 of functions $U_0(x, y)$ having bounded mixed derivatives in Ω to the fourth order. We have the following formula for A_{mn}

$$A_{mn} = \frac{\int_0^{l_1} \int_0^{l_2} U_0(x, y) X_m(x) Y_n(y)\, dx dy}{\|X_m\|_{L_2}^2 \|Y_n\|_{L_2}^2}. \tag{7}$$

By integrating by parts four times in succession in both variables x, y, we hence obtain the following estimate from (7) in the mentioned class C_4

$$|A_{mn}| < \frac{M}{\lambda_m^2 \mu_n^2}, \tag{8}$$

where M is a constant dependent on the maxima of the moduli of the mixed derivatives $\|U_{0xy}\|C$, $\|U_{0xxy}\|C$, $\|U_{0xyy}\|C$, $\|U_{0xxyy}\|C$. It is known [9] that as m, n $\to \infty$ λ_m^2 and μ_n^2 have the asymptotic representation

$$\lambda_m^2 \sim \left[\frac{\pi}{l_1}(m-1)\right]^2, \; \mu_n^2 \sim \left[\frac{\pi}{l_2}(n-1)\right]^2. \tag{9}$$

Now let us consider the function of a complex variable $\psi_m(z) = \sum_{n=1}^{\infty} A_{mn} \exp(-a^2(\lambda_m^2 + \mu_n^2) z) \sin \omega_n$ in the domain Re z > 0. This function is the analytic continuation of the function $\psi_m(t) \in C_{[t_1, t_2]}$ with segment $[t_1, t_2]$ in the right half-plane. Here $\psi_m(z)$ is represented by a series of analytic functions uniformly convergent in any closed subdomain of the domain Re z > 0 (from

(8)); hence [10], it is the unique analytic continuation of the function $\psi_m(t)$ in the domain Re $z > 0$.

Let us consider $\psi_m(z)$ for $z = t > 0$. It follows from (8)-(9) that

$$|\psi_m(t)| = \Big|\sum_{n=1}^{\infty} A_{mn}\exp(-a^2(\lambda_m^2+\mu_n^2)t)\sin\omega_n\Big| \leqslant \sum_{n=1}^{\infty}|A_{mn}|\exp(-a^2\lambda_m^2 t) < M\lambda_m^{-2}\Big(\sum_{n=1}^{\infty}\mu_n^{-2}\Big).$$

Let $C_n = \mu_n^{-2}$, $C_n = O(n^{-2})$ as $n \to \infty$, therefore, $\sum_{n=1}^{\infty} C_n$ converges, hence by the Weierstrass criterion, the series $\sum_{n=1}^{\infty} A_{mn}\exp(-a^2(\lambda_m^2+\mu_n^2)t)\sin\omega_n$ converges uniformly for $0 \leqslant t < \infty$.

Let us apply the Laplace transform to $\psi_m(t)$ (because of the uniform convergence in [0, ∞] integration term by term is possible [11]):

$$\psi_m^*(p) = \int_0^{\infty}\exp(-pt)\psi_m(t)\,dt = \sum_{n=1}^{\infty} A_{mn}\sin\omega_n\ (p+a^2(\lambda_m^2+\mu_n^2))^{-1},\ \operatorname{Re} p > -a^2(\lambda_m^2+\mu_1^2). \tag{10}$$

Let us examine series (10) obtained in the whole complex plane. The terms of this series are analytic functions in the whole plane, with the exception of the points $p_n = -a^2(\lambda_m^2 + \mu_n^2)$, $n = 1, 2, \ldots, \infty$, and the series converges uniformly (because of (8)) in any finite part of the plane not containing these points. Therefore [10], the sum of the series (10) is an analytic function in the whole complex plane with the exception of the points $p_n = -a^2(\lambda_m^2 + \mu_n^2)$. Since for Re $p > -a^2(\lambda_m^2 + \mu_1^2)$ this sum agrees with $\psi_m^*(p)$, then it yields a unique analytic continuation in the whole complex plane except for the singular points p_n, which are isolated first-order poles. Let us now fix a certain number n and let us take the Cauchy integral of $\psi_m^*(p)$ over a circle of small radius ρ around the points p_n. Then

$$\oint_{C_\rho}\psi_m^*(p)\,dp = 2\pi i A_{mn}\sin\omega_n,\ \sin\omega_n \neq 0. \tag{11}$$

It follows from (11) that A_{mn} is determined uniquely for arbitrary m and n, and therefore $U_0(x, y) = \sum_{m,n=1}^{\infty} A_{mn}X_m(x)Y_n(y)$, i.e., uniqueness of the solution of the inverse problem holds.

3. Let us again consider problem (1)-(2). Now, let the temperature on the surface measured similarly to [6] by a certain sensor moving according to a linear law be known as the input data. The following function is therefore known

$$\varphi(t) = U(x(t), 0, t),\ x(t) = \alpha t,\ 0 < t_1 \leqslant t \leqslant t_2,$$
$$0 \leqslant \alpha t_1 < \alpha t_2 < l_1. \tag{12}$$

Let us prove the uniqueness of restoring the initial temperature $U_0(x, y)$ by means of $\varphi(t)$. This latter is represented by the series

$$\varphi(t) = \sum_{m,n=1}^{\infty} A_{mn}\exp(-a^2(\lambda_m^2+\mu_n^2)t)\sin\omega_n\ \sin(\lambda_m\alpha t+\omega_m),\ \sin\omega_m \neq 0,\ 0 < t_1 \leqslant t \leqslant t_2. \tag{13}$$

Let us examine the function of the complex variable

$$\varphi(z) = \sum_{m,n=1}^{\infty} A_{mn}\exp(-a^2(\lambda_m^2+\mu_n^2)z)\sin\omega_n\sin(\lambda_m\alpha z+\omega_m) \tag{14}$$

in the domain Re $z > 0$. Series (14) of analytic functions converges uniformly in any closed subdomain of the domain Re $z > 0$ (the series $\sum_{m,n=1}^{\infty}|A_{mn}| = M\sum_{m,n=1}^{\infty} m^{-2}n^{-2}$ because of (9), and such a double series converges). Hence, there results [10] that $\varphi(z)$ is a unique analytic continuation of $\varphi(t) \in C_{[t_1, t_2]}$ in the domain Re $z > 0$. We examine (14) for $z = t > 0$. It follows from (8) that the series (13) converges uniformly for $0 \leqslant t < \infty$. We apply the Laplace transform to $\varphi(t)$

$$\varphi^*(p) = \sum_{m,n=1}^{\infty} A_{mn} \frac{p + a^2(\lambda_m^2 + \mu_n^2)\sin\omega_n + \lambda_m\alpha\cos\omega_n}{[p + a^2(\lambda_m^2 + \mu_n^2)]^2 + \lambda_m^2\alpha^2} \sin\omega_n, \ \mathrm{Re}\, p > -a^2(\lambda_1^2 + \mu_1^2). \tag{15}$$

Taking (8)-(9) into account and executing the same reasoning as in Sec. 2, we obtain that series (15) determines an analytic function on the whole plane with the exception of the points $p_{mn}^{1,2} = -a^2(\lambda_m^2 + \mu_n^2) \pm i\lambda_m\alpha$, which are isolated first-order poles. As is easy to see, a mutually one-to-one correspondence exists between the points p_{mn}^1 and p_{mn}^2 and the numbers (m, n), and therefore there are no multiple roots. Let us set arbitrary m and n and let us take the Cauchy integral of $\varphi^*(p)$ over a circle of small radius ρ with center at p_{mn}^1:

$$\oint_{C_\rho} \varphi^*(p)\, dp = A_{mn} \sin\omega_n q(m, n, \alpha), \tag{16}$$

$$q(m, n, \alpha) = \frac{a^2(\lambda_m^2 + \mu_n^2)(\sin\omega_n - 1) + \lambda_m\alpha\cos\omega_n - i\lambda_m\alpha}{\lambda_m\alpha},$$

where $q(m, n, \alpha) \neq 0$ for any m, n, α, since Im $q = -\lambda_m\alpha \neq 0$.

The A_{mn} is determined uniquely from (16), and therefore

$$U_0(x, y) = \sum_{m,n=1}^{\infty} A_{mn} X_m(x) Y_n(y).$$

4. On the basis of the facts established in this paper about the uniqueness of the inverse problems, stable numerical algorithms can be constructed for the restoration of the initial temperature field and, as a result, a number of control problems for technological processes can be solved.

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